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Approximation on [0, ∞) by Reciprocals of Polynomials with Nonnegative Coefficients*

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A complete theory of best uniform approximation to positive functions decaying to zero on [0, ∞) by reciprocals of polynomials with nonnegative coefficients is presented.

1. INTRODUCTION

Let $C_0^+(X)$ denote the class of all real-valued continuous functions defined on $X \subseteq [0, \infty)$, where X is closed, $f(x) > 0$ on X and $f(x) \to 0$ as $x \to \infty$ (in X) if X is unbounded. Let $K(X) = \{p \in \Pi_n : p(x) > 0 \quad \forall x \in X$ and $p^{(j)}(0) \geqslant 0$, $j = 0, 1, \ldots, n\}$, where Π_n denotes the class of all real algebraic polynomials of degree $\leqslant n$. Thus, K consists of positive polynomials with nonnegative coefficients (we suppress the X whenever possible). We give existence, characterization and (strong) uniqueness results for the problem of best approximating functions $f \in C_0^+[0, \infty)$ by reciprocals of elements of K.

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In an earlier paper, Reddy and Shisha [8] showed that the closure of the reciprocals of all polynomials having nonnegative coefficients on $[0, \infty)$ is the set of all reciprocals of entire functions with nonnegative Taylor coefficients.

Although our primary interest is $[0, \infty)$, the theory is developed for X a closed subset of $[0, \infty)$. The assumption that X is closed guarantees that $\|f\|_X = \max\{|f(x)|: x \in X\} < \infty$ for each $f \in C_0^+(X)$.

In Section 2, we begin by establishing an existence theorem. In Section 3, two characterization results are given assuming X is compact. These characterizations are based upon certain linear functionals in Π_n^*, the dual of Π_n. In Section 4 strong uniqueness is shown to hold when X is compact. In Section 5 it is shown that obtaining the best approximation to $f \in C_0^+[0, \infty)$ from $K[0, \infty)$ is equivalent to finding the best approximation on $[0, b]$ from $K[0, b]$, where b may be determined constructively. Combining these results with the results of the previous two sections establishes characterization and uniqueness for the $[0, \infty)$ problem. In Section 6 this theory is then extended to X, a closed subset of $[0, \infty)$, and a discretization result is established. Finally, in Section 7 some numerical examples are given

2. EXISTENCE

We begin by developing an existence theory for this problem. Note that this requires $\|f\|_X < \infty$ and also requires a little care as it might be possible for p to become unbounded near where $f(x)$ is "small."

THEOREM 1 (EXISTENCE). *Let $f \in C_0^+(X)$, where X is a closed subset of $[0, \infty)$. Then there a $p^* \in K$ such that*

$$\left\| f - \frac{1}{p^*} \right\|_X = \inf \left\{ \left\| f - \frac{1}{p} \right\|_X : p \in K \right\}.$$

Proof. If $n = 0$, then $1/p^*$ is best with $1/p^* = \frac{1}{2}(\|f\|_X + \inf_X |f(x)|)$, where we have used the fact that $\|f\|_X < \infty$. Therefore, assume $n \geqslant 1$. Without loss of generality we may assume $\operatorname{card}(X) \geqslant n+2$. Let $\rho = \inf_{p \in K} \|f - 1/p\|_X$ and let $\{p_l\}_{l=1}^{\infty} \subseteq K$ be such that $\|f - 1/p_l\|_X \searrow \rho$. Setting $p_l(x) = \sum_{i=0}^n a_{li} x^i$, if we can show that $\{a_{li}\}$ is bounded, then by using subsequences (relabelled) we can find $p^*(x) = \sum_{i=0}^n a_i^* x^i$ with $a_{li} \to a_i^*$, so $a_i^* \geqslant 0$, $0 \leqslant i \leqslant n$. Furthermore, we must have $p^*(x) \geqslant 1/(f(x) + \rho + 1)$, $\forall x \in X$ and $\|f - 1/p^*\|_X \leqslant \rho$, so $1/p^*$ is best.

Therefore, let us assume that $\{a_{li}\}$ is unbounded so (taking a subsequence of $\{p_l\}$, if necessary) $\max_i a_{li} \to \infty$ as $l \to \infty$. Define $q_l(x) = (\max_i a_{li})^{-1} p_l(x) = \sum_{i=0}^n b_{li} x^i$. Again, using subsequences if necessary, we can find $q(x)=\sum_{i=0}^{n} b_i x^i$ with $b_{li} \to b_i$, $0 \leqslant i \leqslant n$, and $\max_i b_i = 1$, $b_i \geqslant 0$, $0 \leqslant i \leqslant n$. Hence $q(x) > 0$ for $x > 0$. For $x \in X \setminus \{0\}$ we have $p_l(x) = (\max_i a_{li})\, q_l(x) \to \infty$ as $l \to \infty$. Therefore, since $1/p_l(x) \to 0$ as $l \to \infty$ and

$$\left| f(x) - \frac{1}{p_l(x)} \right| \leqslant \left\| f - \frac{1}{p_l} \right\| \downarrow \rho,$$

taking the limit as $l \to \infty$ yields $0 < f(x) \leqslant \rho$ (thus $\rho > 0$), $x \in X \setminus \{0\}$. But this leads to a contradiction since $p(x) = 2/\rho$ satisfies $\|f - 1/p\| \leqslant \rho/2$ if 0 is not an isolated point of X, whereas $p(x) = Mx + (f(0))^{-1}$ satisfies $\|f - 1/p\| < \rho$ for M sufficiently large if 0 is an isolated point of X. ∎

In closing this section we observe that if X is unbounded and $n \geqslant 1$ then the best reciprocal approximation to $f \in C_0^+(X)$ from $K(X)$ is not a constant. This is easily seen by observing that the best reciprocal constant approximation is $c^* = 2/\|f\|_X$ and that for a proper choice of ε_1, $\varepsilon_2 > 0$, $p^*(x) = \varepsilon_2 x + (c^* - \varepsilon_1)$ will belong to K and satisfy $\|f - 1/c^*\|_X > \|f - 1/p^*\|_X$.

3. Characterization

In this section we shall assume that X is compact and establish both a "zero in the convex hull" type of characterization and a generalized alternation characterization. In both cases, these results are analogous to the characterization for approximation as developed in [2]. In order to obtain these results, we use specific linear functionals in Π_n^*, the dual space of Π_n with the uniform topology. Basically, two types of linear functionals play a crucial role. They are point evaluations $e_x \in \Pi_n^*$, where $e_x(g) = g(x)$, $\forall g \in C(X)$, $x \in X$, and derivative evaluations at zero $e_0^j \in \Pi_n^*$, where $e_0^j(p) = p^{(j)}(0)$, $\forall p \in \Pi_n$, $0 \leqslant j \leqslant n$.

Fix $f \in C_0^+(X)$ and $p \in K$. Then we say that $e \in \Pi_n^*$ is an *extreme point* for f and p if either

(i) $e \equiv e_x$ for some $x \in X$ and $|e_x(f - 1/p)| = \|f - 1/p\|_X$, or

(ii) $e \equiv e_0^j$ for some j, $0 \leqslant j \leqslant n$ and $e_0^{(j)}(p) = 0$.

We denote the complete set of all extreme points for f and p by X_p, as usual. In addition, we define the sign of an extreme point $\sigma(e)$ by

(1) $\sigma(e) = \operatorname{sgn}(f(x) - 1/p(x))$ if $e \equiv e_x$, or

(2) $\sigma(e_0^j) = (-1)^{j+1}$.

We observe that it is not possible for both e_0 and e_0^0 to belong to the extreme set of f and p. In fact, $e_0 \in X_p$ can occur only if $0 \in X$ and $e_0^0 \in X_p$ can occur only if $0 \notin X$ (since $0 \in X$ implies that $p(0) > 0$, as $p \in K$).

We note that any k distinct extreme points for f and p with $k \leqslant n+1$ are linearly independent. Also, any set of $n+2$ extreme points for f and p will be linearly dependent as Π_n^* has dimension $n+1$. Finally, we observe that due to the continuity of f and p on X, it follows that X_p is a compact subset of Π_n^*. Let

$$U = \{-e_0^j : e_0^j \in X_p\} \cup \{\sigma(e_x)\, e_x : e_x \in X_p\}.$$

Then we have the following "zero in the convex hull" characterization theorem.

THEOREM 2. *Let $f \in C_0^+(X)$ be such that $1/f \notin K$. Then $p^* \in K$ gives a best reciprocal approximation to f from K on X (compact) iff the zero of Π_n^* belongs to the convex hull, $H(U)$, of U corresponding to X_{p^*}. Furthermore, the convex combination will always consist of precisely $n+2$ nonzero terms.*

Proof. ($\Leftarrow$) By contradiction. Therefore, we assume that $p^* \in K$ does not give a best approximation to f. Then, $\exists p \in K \ni \|f - 1/p\| < \|f - 1/p^*\|$. Let $p(x) = \sum_{i=0}^n a_i x^i$ and set $p_\varepsilon(x) = \sum_{i=0}^n (a_i + \varepsilon)\, x^i$. Since X is compact, we select $\varepsilon > 0$ sufficiently small so that $\|f - 1/p_\varepsilon\| < \|f - 1/p^*\|$. Then, for $e_0^j \in X_{p^*}$ we have that $-e_0^j(p_\varepsilon - p^*) < 0$. Also, for $e_x \in X_{p^*}$, we have from the inequality

$$\sigma(e_x)\left(\frac{1}{p^*(x)} - \frac{1}{p_\varepsilon(x)}\right) < 0,$$

that $\sigma(e_x)\, e_x(p_\varepsilon - p^*) < 0$. Thus, the system of linear inequalities $e(p) < 0$, $e \in U$, is consistent. Since U is compact (as is X_{p^*}) we have, by the Theorem on Linear Inequalities (see, e.g., [3, p. 19]) (identifying Π_n^* and Π_n with R^n), that zero does not belong to the convex hull of U. This is a contradiction establishing the desired result.

($\Rightarrow$) By contradiction. Therefore, we assume $0 \notin H(U)$. Again, by the Theorem on Linear Inequalities, we have that $\exists q \in \Pi_n$ such that $-e_0^j(q) < 0$ for all $e_0^j \in X_{p^*}$ and $\sigma(e_x)\, e_x(q) < 0$ for all $e_x \in X_{p^*}$. Set $p_\varepsilon = p^* + \varepsilon q$, where $\varepsilon > 0$ is chosen sufficiently small so that $p_\varepsilon(x) > 0$ for all $x \in X$. Now, for $e_0^j \in X_{p^*}$ we have that $q^{(j)}(0) > 0$ so that $p_\varepsilon^{(j)}(0) > 0$. By taking $\varepsilon > 0$ smaller, if necessary, we can also guarantee that $p_\varepsilon^{(j)}(0) > 0$ for all j, $0 \leqslant j \leqslant n$, such that $e_0^j \notin X_{p^*}$ since $p^{*(j)}(0) > 0$ for these indices. Hence $p_\varepsilon \in K$.

We now claim that for $\varepsilon > 0$ (chosen smaller yet, if necessary), we must have that $\|f - 1/p_\varepsilon\| < \|f - 1/p^*\|$ giving the desired contradiction. A standard compactness argument gives this result since at the positive extremals e_x (i.e., $\sigma(e_x) = 1$) we have that $q(x) < 0$ so that $1/p^*(x) < 1/p_\varepsilon(x)$ and at the negative extremals $1/p^*(x) > 1/p_\varepsilon(x)$.

Finally, since Π_n^* is $n+1$ dimensional, we have that the zero in the convex hull result will hold with $s \leqslant n+2$ terms. In order to see that it is not possible for this to hold with less than $n+2$ terms, we simply note that for a set S of $s < n+2$ distinct elements of X_{p^*} we can always find $p \in \Pi_{s-1}$ for which $e_0^j(p) = -1$ if $e_0^j \in S$ and $e_x(p) = \sigma(e_x)$ if $e_x \in S$. This follows from the fact that the Hermite–Birkhoff problem associated with these equations is poised (i.e., all supported blocks are even, see [1]).

We now turn to developing our generalized alternation theorem. To this end, fix f and let $p \in K$. We say that $\{e_0^{j_\nu}\}_{\nu=1}^s \cup \{e_{x_\mu}\}_{\mu=s+1}^k \subset X_p$ is an *alternant of length k for* $f - 1/p$ provided $n \geqslant j_1 > j_2 > \cdots > j_s \geqslant 0$; $x_{s+1} < x_{s+2} < \cdots < x_k$ with

(1) $j_\nu - j_{\nu+1}$ an odd integer for $\nu = 1, 2, ..., s-1$ (if $s \leqslant 1$, then this requirement is vacuous),

(2) $\sigma(e_{x_{s+1}}) = (-1)^{j_s}$ (if $s = 0$, or $s = k$, then this requirement is vacuous), and

(3) $\sigma(e_{x_\mu}) = -\sigma(e_{x_{\mu+1}})$, $\mu = s+1, ..., k-1$ (vacuous if $k \leqslant s+1$). Thus, (1)–(3) imply that if $\{e_l\}_{l=1}^k = \{e_0^{j_\nu}\}_{\nu=1}^s \cup \{e_{x_\mu}\}_{\mu=s+1}^k$, listed in this order, then $\sigma(e_{l+1}) = -\sigma(e_l)$ for $l = 1, ..., k-1$.

With this definition, we have

THEOREM 3. *Suppose* $f \in C_0^+(X)$ *and* $1/f \notin K$. *Then* $p^* \in K$ *gives a best reciprocal approximation to f from K on X (compact) iff* $f - 1/p^*$ *has an alternant of length* $n+2$.

Proof. The method of proof is to show that this alternant is precisely a basis for the "zero in the convex hull" result of Theorem 2. The specific proof given here is patterned after one given by B. Chalmers [2, Theorem 2, Section 4].

($\Leftarrow$) Suppose that p^* gives a best reciprocal approximation to f from K on X. Then, there exist positive constant $\lambda_1, ..., \lambda_{n+2}$ with $\sum_{i=1}^{n+2} \lambda_i = 1$, and a set of $n+2$ distinct extremals in X_{p^*}, $\{e_0^{j_\nu}\}_{\nu=1}^s \cup \{e_{x_\mu}\}_{\mu=s+1}^{n+2}$ ordered as above (*i.e.*, $n \geqslant j_1 > j_2 > \cdots > j_s \geqslant 0$, $x_{s+1} < x_{s+2} < \cdots < x_{n+2}$) such that

$$\sum_{\nu=1}^{s} \lambda_\nu(-e_0^{j_\nu}) + \sum_{\mu=s+1}^{n+2} \lambda_\mu \sigma(e_{x_\mu})\, e_{x_\mu} = 0 \tag{1}$$

in Π_n^*. Now set $J = \{j_s, j_{s-1}, ..., j_1\}$ and $I = \{0, 1, ..., n\} \backslash J$. Now apply the linear combination (1) to the functions x^{j_k}, $k = s, s-1, ..., 1$, which yields

$$\sum_{\mu=s+1}^{n+2} \lambda_\mu \sigma(e_{x_\mu})\, x_\mu^{j_k} = (j_k!)\, \lambda_{j_k}, \qquad k = s, s-1, ..., 1. \tag{2}$$

Applying (1) to the function x^m, $m \in I$, gives

$$\sum_{\mu=s+1}^{n+2} \lambda_\mu \sigma(e_{x_\mu})\, x_\mu^m = 0, \qquad m \in I. \tag{3}$$

Note that (3) consists of precisely $n+1-s$ equations and $n+2-s$ coefficients. Now, using the fact that $\det[(x_i^{\rho_j})_{i,j=1}^l] > 0$ for $0 < x_1 < \cdots < x_l < \infty$ and $-\infty < \rho_1 < \rho_2 < \cdots < \rho_l < \infty$ (see, e.g., [4, p. 9]) and Cramer's rule we have by standard techniques (see, e.g., [3, p. 74])

$$\operatorname{sgn} \lambda_\mu \sigma(e_{x_\mu}) = -\operatorname{sgn} \lambda_{\mu+1} \sigma(e_{x_{\mu+1}}), \qquad \mu = s+1, \ldots, n+1,$$

or

$$\sigma(e_{x_\mu}) = -\sigma(e_{x_{\mu+1}}), \qquad \mu = s+1, \ldots, n+1, \text{ as } \lambda_i > 0, \ \forall i.$$

Next, in system (2), observe that the functions $\phi_1(t) = x_{s+1}^t$, $\phi_2(t) = x_{s+2}^t, \ldots, \phi_{n+2-s}(t) = x_{n+2}^t$ (use $\phi_2, \phi_3, \ldots, \phi_{n+2-s}$ if $x_{s+1} = 0$) form a Chebyshev system for $t \in [0, \infty)$. Thus,

$$F(t) = \sum_{\mu=s+1}^{n+2} \{\lambda_\mu \sigma(e_{x_\mu})\} \, x_\mu^t$$

can have at most $n+1-s$ zeros in $[0, \infty)$ counting a zero at which $F(t)$ does not change sign as two zeros (for $x_{s+1} = 0$, use $F(t) = \sum_{\mu=s+2}^{n+2} \{\lambda_\mu \sigma(e_{x_\mu})\} \, x_\mu^t$ which can have only $n-s$ zeros in $[0, \infty)$. Note that $F(0) = -\lambda_{s+1} \sigma(e_{x_{s+1}}) \neq 0$. This is the equation of (3) corresponds to $m = 0$. Recall that $0 \in X$ implies that $j_s > 0$).

Now $F(t)$ vanishes at $t = m$, $m \in I$, for a total of $n+1-s$ points. (For the case $x_{s+1} = 0$, $F(t)$ vanishes at $t = m$, $m \in I$, $m \neq 0$, for $n-s$ points.) Thus, each point of $I \backslash \{0\}$ must be a point where $F(t)$ changes sign and $F(t)$ can have no additional positive zeros. Now, since $(j_k!) \, \lambda_{j_k} > 0$ for $k = s$, $s-1, \ldots, 1$ we see that for $j_k \in J$, j_{k+1} and j_k must have an even number of elements of I between them (0 is allowed). That is, $j_k - j_{k+1}$ must be an odd integer for $k = 1, \ldots, s-1$.

Finally, define $p \in \Pi_n$ by $p^{(j)}(0) = 0$, $j \in J \backslash \{j_s\}$, $p(x_\mu) = 0$, $\mu = s+2, \ldots, n+2$ and $p(x_{s+1}) = 1$, where $p(x) = \sum_{i=0}^n a_i x^i$. Observing that $\{0, 1, \ldots, j_s - 1\} \subset I$, we shall enumerate $I \cup \{j_s\}$ by $I \cup \{j_s\} \equiv \{0, 1, \ldots, j_s, l_{s+1}, \ldots, l_{n+1-s}\}$, where $j_s < l_{s+1} < \cdots < l_{n+1-s} \leqslant n$. Then p satisfies the system

$$\sum_{m=0}^{l_{n+1-s}} a_m x_\mu^m = \delta_{s+1,\mu}, \qquad \mu = s+1, \ldots, n+2. \tag{4}$$

Solving for a_{j_s} by Cramer's rule and using the fact that $\det[(x_i^{\rho_j})_{i,j=1}^l] > 0$ for $0 < x_1 < \cdots < x_l < \infty$, $-\infty < \rho_1 < \cdots < \rho_l < \infty$ again, we see, after j_s column interchanges in the numerator determinant, that $\operatorname{sgn}(a_{j_s}) = (-1)^{j_s}$. Now, applying (1) to p we find that

$$-\lambda_{j_s}(j_s!) \, a_{j_s} + \lambda_{s+1} \sigma(e_{x_{s+1}}) = 0,$$

or that $\sigma(e_{x_{s+1}}) = (-1)^{j_s}$. This shows that the extreme points of the "zero in the convex hull" characterization form an alternant of length $n+2$ for $f - 1/p^*$.

($\Rightarrow$) Conversely, let $\{e^{j_\nu}\}_{\nu=1}^{s} \cup \{e_{x_\mu}\}_{\mu=s+1}^{n+2}$ be an alternant of length $n+2$ for $f - 1/p^*$. Then, since Π_n^* is $n+1$ dimensional and any $n+1$ of the above extremals form a basis for Π_n^* we have that $\exists$ constants $\theta_1,..., \theta_{n+2}$, all not zero, such that

$$\sum_{\nu=1}^{s} \theta_\nu(-e_0^{j_\nu}) + \sum_{\mu=s+1}^{n+2} \theta_\mu \sigma(e_{x_\mu})\, e_{x_\mu} = 0 \tag{5}$$

in Π_n^*. Define J and I as above and apply (5) to x^m, $m = 0, 1,..., n$ to obtain

$$\sum_{\mu=s+1}^{n+2} \theta_\mu \sigma(e_{x_\mu})\, x_\mu^{j_k} = \theta_{j_k}(j_k!), \qquad k = s, s-1,..., 1, \tag{6}$$

and

$$\sum_{\mu=s+1}^{n+2} \theta_\mu \sigma(e_{x_\mu})\, x_\mu^m = 0, \qquad m \in I. \tag{7}$$

Now, as above, (7) implies that $\operatorname{sgn}(\theta_\mu \sigma(e_{x_\mu})) = -\operatorname{sgn}(\theta_{\mu+1} \sigma(e_{x_{\mu+1}}))$, $\mu = s+1,..., n+1$. Since $\sigma(e_{x_\mu}) = -\sigma(e_{x_{\mu+1}})$ for $\mu = s+1,..., n+1$ we have that $\operatorname{sgn} \theta_\mu = \operatorname{sgn} \theta_{\mu+1}$, $\mu = s+1,..., n+1$. Next, for the special function p defined by $p^{(j_k)}(0) = 0$, $k = s-1,..., 1$, $p(x_\mu) = 0$, $\mu = s+2,..., n+2$ and $p(x_{s+1}) = 1$, we get, after applying (5) to this p, that $\theta_{s+1}\sigma(e_{x_{s+1}}) = \theta_s p^{(j_s)}(0)$. Since $\sigma(e_{x_{s+1}}) = (-1)^{j_s}$ and $\operatorname{sgn} p^{(j_s)}(0) = (-1)^{j_s}$ from above, we have that $\operatorname{sgn}(\theta_{s+1}) = \operatorname{sgn}(\theta_s)$. Finally, by repeating the $F(t)$ argument appearing in the first half of this proof we have that $\operatorname{sgn} \theta_\nu = \operatorname{sgn} \theta_{\nu-1}$ for $\nu = s, s-1,..., 2$ as desired. Thus, $\theta_i \neq 0$, $i = 1, 2,..., n+2$, and all are of the same sign. Hence (using a suitable normalization), we have that the zero of Π_n^* belongs to the convex hull of U, U corresponding to X_{p^*}, as above (in fact, we know a specific convex combination from U for 0). Thus, $p^* \in K$ gives a best reciprocal approximation to f from K on X as desired. ∎

We observe that in an alternant of length $n+2$ for $f - 1/p^*$, we must have $s \leqslant n$, so that there will always exist at least two standard extremals and normal alternation between them; if p^* is not a constant and $0 \in X$ then $s \leqslant n-1$, so there will be at least three standard extremals and normal alternation between them.

4. Uniqueness

Best approximations in our setting are unique; in fact, the zero in the convex hull theorem enables us to prove strong uniqueness. Lipschitz continuity of the best approximation operator then follows as in [3, p. 82]. In this section we shall write $\|\cdot\|$ for $\|\cdot\|_X$.

THEOREM 4. *Let $f \in C_0^+(X)$, where X is compact, and let $p^* \in K$ satisfy $\|f - 1/p^*\| = \inf_{p \in K} \|f - 1/p\|$. Then there exists a positive constant $\gamma = \gamma(f)$ such that*

$$\left\|f - \frac{1}{p}\right\| \geqslant \left\|f - \frac{1}{p^*}\right\| + \gamma \left\|\frac{1}{p} - \frac{1}{p^*}\right\|$$

for all $p \in K$.

Proof. Without loss of generality we may assume $\|f - 1/p^*\| > 0$, since otherwise the theorem holds with $\gamma = 1$. For $p \in K$, $p \not\equiv p^*$, define

$$\gamma(p) = \frac{\left\|f - \frac{1}{p}\right\| - \left\|f - \frac{1}{p^*}\right\|}{\left\|\frac{1}{p} - \frac{1}{p^*}\right\|}.$$

Assume (by way of contradiction) that there exist a sequence $\{p_k\} \subseteq K$, $p_k \not\equiv p^*$, with $\gamma(p_k) \to 0$. Then $\|1/p_k\|$ is bounded (otherwise $\gamma(p_k) \not\to 0$), and thus $\|f - 1/p_k\| - \|f - 1/p^*\| \to 0$ (otherwise $\gamma(p_k) \not\to 0$), so from the proof of Theorem 1 we have that $\|p_k\|$ must be bounded. By Theorem 2 there is a set of $n + 2$ distinct extremals

$$U = \{e_0^{j_\nu}\}_{\nu=1}^{s} \cup \{e_{x_\mu}\}_{\mu=s+1}^{n+2} \subseteq X_{p^*}$$

and a set $\{\lambda_i\}_{i=1}^{n+2}$ of positive constants such that

$$\sum_{\nu=1}^{s} \lambda_\nu(-e_0^{j_\nu}) + \sum_{\mu=s+1}^{n+2} \lambda_\mu \sigma(e_{x_\mu})\, e_{x_\mu} = 0 \in \Pi_n^*.$$

Now let $p \in K$ satisfy

$$-e_0^{j_\nu}(p) \leqslant 0, \qquad \nu = 1,\dots, s,$$

and

$$\sigma(e_{x_\mu})\, e_{x_\mu}(p) \leqslant 0, \qquad \mu = s+1,\dots, n+2.$$

Then from

$$\sum_{\nu=1}^{s} \lambda_\nu(-e_0^{j_\nu}(p)) + \sum_{\mu=s+1}^{n+2} \lambda_\mu \sigma(e_{x_\mu}) e_{x_\mu}(p) = 0$$

and the fact that $\lambda_i > 0$ for $i = 1,..., n+2$, we get

$$e_0^{j_\nu}(p) = 0, \qquad \nu = 1,..., s,$$

and

$$e_{x_\mu}(p) = 0, \qquad \mu = s+1,..., n+2.$$

But any $n+1$ of these conditions imply that $p \equiv 0$, since the associated Hermite–Birkhoff interpolation problem is poised. Thus, if $p \in K$ satisfies $p \not\equiv 0$ and $-e_0^{j_\nu}(p) \leqslant 0$, $\nu = 1,..., s$, then for some ω with $s+1 \leqslant \omega \leqslant n+2$ we must have $\sigma(e_{x_\omega}) e_{x_\omega}(p) > 0$. Let

$$c = \inf\{ \max_{s+1 \leqslant \mu \leqslant n+2} \sigma(e_{x_\mu}) p(x_\mu):$$
$$p \in K, \|p\| = 1 \text{ and } -e_0^{j_\nu}(p) \leqslant 0, \nu = 1,..., s\} > 0.$$

Then for all $\mu = s+1,..., n+2$, we have

$$\begin{aligned}
\gamma(p_k) \left\| \frac{1}{p_k} - \frac{1}{p^*} \right\| &= \left\| f - \frac{1}{p_k} \right\| - \left\| f - \frac{1}{p^*} \right\| \\
&\geqslant \sigma(e_{x_\mu}) \left(f(x_\mu) - \frac{1}{p_k(x_\mu)} \right) - \sigma(e_{x_\mu}) \left(f(x_\mu) - \frac{1}{p^*(x_\mu)} \right) \\
&= \sigma(e_{x_\mu}) \left(\frac{1}{p^*(x_\mu)} - \frac{1}{p_k(x_\mu)} \right) \\
&= \sigma(e_{x_\mu}) \frac{p_k(x_\mu) - p^*(x_\mu)}{p^*(x_\mu)\, p_k(x_\mu)} \\
&= \frac{\| p_k - p^* \|}{p^*(x_\mu)\, p_k(x_\mu)} \left[\sigma(e_{x_\mu}) \cdot \frac{p_k(x_\mu) - p^*(x_\mu)}{\| p_k - p^* \|} \right].
\end{aligned}$$

So for some $\omega = s+1,..., n+2$, we have

$$\gamma(p_k) \left\| \frac{1}{p_k} - \frac{1}{p^*} \right\| \geqslant \frac{\| p_k - p^* \|}{p^*(x_\omega)\, p_k(x_\omega)} \cdot c.$$

Now for each k select $y_k \in X$ such that

$$\left| \frac{1}{p_k(y_k)} - \frac{1}{p^*(y_k)} \right| = \left\| \frac{1}{p_k} - \frac{1}{p^*} \right\|.$$

Then,

$$\left\|\frac{1}{p_k}-\frac{1}{p^*}\right\| \leqslant \frac{\|p_k-p^*\|}{p_k(y_k)\,p^*(y_k)}$$

so that

$$\gamma(p_k)\frac{\|p_k-p^*\|}{p^*(y_k)p_k(y_k)} \geqslant \frac{\|p_k-p^*\|}{p^*(x_\omega)\,p_k(x_\omega)}\cdot c.$$

Hence,

$$\gamma(p_k) \geqslant \frac{p^*(y_k)\,p_k(y_k)}{p^*(x_\omega)\,p_k(x_\omega)}\cdot c \nrightarrow 0,$$

as $\|p_k\|$ and $\|1/p_k\|$ are bounded independent of k and X is compact. This gives us our desired contradiction, completing the proof. ▮

5. Approximation on $[0, \infty)$

We now state and prove a central result which shows that for $n \geqslant 1$, approximation on $[0, \infty)$ with reciprocals of elements of K is completely equivalent to approximation on $[0, b]$ for some $b > 0$. This result allows us to apply the theory of the previous sections to this problem. Also, this proof can be made contructive, giving a procedure for calculating b.

THEOREM 5. *Let $f \in C_0^+[0, \infty)$ and assume $n \geqslant 1$. Then there exists $b > 0$, $p^* \in K[0, \infty) \equiv K$ such that*

$$\left\|f-\frac{1}{p^*}\right\|_{[0,b]} = \inf_{p\in K}\left\|f-\frac{1}{p}\right\|_{[0,b]} = \inf_{p\in K}\left\|f-\frac{1}{p}\right\|_{[0,\infty)}$$
$$= \left\|f-\frac{1}{p^*}\right\|_{[0,\infty)} = \lambda_\infty.$$

Proof. We can assume $1/f \notin K$. For each $0 < b \leqslant \infty$, choose $p_b \in K$ satisfying

$$\left\|f-\frac{1}{p_b}\right\|_{[0,b]} = \inf_{p\in K}\left\|f-\frac{1}{p}\right\|_{[0,b]} = \lambda_b.$$

Assume p_b cannot serve as p_∞ for all finite positive b. Then by uniqueness of such p_b, p_∞ cannot serve as p_b for any finite positive b. Hence for all $0 < b < \infty$,

$$\left\|f-\frac{1}{p_b}\right\|_{[0,\infty)} > \left\|f-\frac{1}{p_\infty}\right\|_{[0,\infty)} \geqslant \left\|f-\frac{1}{p_\infty}\right\|_{[0,b]} > \left\|f-\frac{1}{p_b}\right\|_{[0,b]}. \tag{8}$$

Then for some $y_b > b$,

$$\left| f(y_b) - \frac{1}{p_b(y_b)} \right| > \left\| f - \frac{1}{p_\infty} \right\|_{[0,\infty)} > \left| f(b) - \frac{1}{p_b(b)} \right| .$$

But $p_b(y_b) \geqslant P_b(b)$ and $\max\{f(x): x \geqslant b\} \to 0$ as $b \to \infty$. We deduce

$$\lim_{b \to \infty} \frac{1}{p_b(b)} = \left\| f - \frac{1}{p_\infty} \right\|_{[0,\infty)} .$$

Write $p_b(x) = \sum_{j=0}^n a_{jb} x^j$. Then if $y > 0$ is given, and $b > y$,

$$\left\| \sum_{j=1}^n a_{jb} x^j \right\|_{[0,y]} = \sum_{j=1}^n a_{jb} \left(\frac{y}{b}\right)^j b^j < \frac{y}{b} p_b(b) \to 0 \qquad \text{as} \quad b \to \infty.$$

Further $a_{0b} \leqslant M$, some $M < \infty$, for all $b > 0$ as $\|f - 1/p_\infty\|_{[0,\infty)} < \frac{1}{2} \|f\|_{[0,\infty)}$. Choose a sequence B of values for b such that as $b \to \infty$ through B, $a_{0b} \to c$. Then we see, as c is independent of y, that

$$\lim_{\substack{b \to \infty \\ b \in B}} \left\| f - \frac{1}{p_b} \right\|_{[0,y]} = \left\| f - \frac{1}{c} \right\|_{[0,y]} \qquad \text{for each } y > 0.$$

Then using the last inequality in (8), we see

$$\left\| f - \frac{1}{c} \right\|_{[0,y]} \leqslant \limsup_{\substack{b \to \infty \\ b \in B}} \left\| f - \frac{1}{p_b} \right\|_{[0,b]} \leqslant \left\| f - \frac{1}{p_\infty} \right\|_{[0,\infty)}$$

for each $y > 0$. We deduce that

$$\left\| f - \frac{1}{c} \right\|_{[0,\infty)} = \left\| f - \frac{1}{p_\infty} \right\|_{[0,\infty)}$$

so that a constant c is a best approximation; as after Theorem 1, this is impossible. ∎

Remark. A constructive proof can be given for calculating b in which at most four best reciprocal approximations need be calculated. A copy of this is available upon request.

COROLLARY. *The best approximation to $f \in C_0^+[0, \infty)$, for $n \geqslant 1$, exists, is unique, and is characterized by the alternation of Theorem* 3.

Note that strong uniqueness need not hold in the $[0, \infty)$ setting. For example, if $n=3$, $p^*(x)=x+1$ is readily seen to be the unique best reciprocal approximation to $f(x)$ by the standard alternating theorem where $f(x)$ is defined to be piecewise linear on $[0, \frac{5}{4}]$ with vertices $(\nu/4, (1+\nu/4)^{-1} - \frac{1}{4}(-1)^\nu)$, $\nu = 0, \ldots, 4$ and $(\frac{5}{4}, (1+\frac{5}{4})^{-1})$. For $x \geqslant \frac{5}{4}$, $f(x)$ is defined to be $(x+1)^{-1}$. Setting $p_k(x) = 1 + x + x^k/k$, one can show that strong uniqueness fails to hold in this case.

6. Discretization Results

Suppose X is a nonvoid closed subset of $[0, \infty)$. Define $|X| = \sup_{x \in [0,\infty)} \inf_{y \in X} |x-y| =$ density of X in $[0, \infty)$. Then we have

THEOREM 6. *If $f \in C_0^+(X)$, $n \geqslant 1$, then there exists a $b > 0$ and a $p^* \in K = K(X)$ such that*

$$\left\| f - \frac{1}{p^*} \right\|_{[0,b] \cap X} = \inf_{p \in K} \left\| f - \frac{1}{p} \right\|_{[0,b] \cap X} = \inf_{p \in K} \left\| f - \frac{1}{p} \right\|_X = \left\| f - \frac{1}{p^*} \right\|_X.$$

Proof. The proof follows the proof of Theorem 5 where each interval is replaced by its intersection with X, and where each point mentioned is in X.

COROLLARY. *The best reciprocal approximation to $f \in C_0^+(X)$ on X, for $n \geqslant 1$, exists, is unique, and is characterized by the alternation of Theorem* 3.

Now, let $n \geqslant 1$, $f \in C_0^+[0, \infty)$ and $1/f \notin K[0, \infty)$. Define λ_b, λ_∞ as in Theorem 5 (note $\lambda_\infty > 0$) and define

$$\lambda_b^X = \inf_{p \in K(X)} \left\| f - \frac{1}{p} \right\|_{[0,b] \cap X},$$

$$\lambda_\infty^X = \inf_{p \in K(X)} \left\| f - \frac{1}{p} \right\|_X,$$

$1/p_b^X =$ best approximation to f on $[0, b] \cap X$ where $p_b^X \in K(X)$,

$1/p_\infty^X =$ best approximation to f on X where $p_\infty^X \in K(X)$,

$1/p_\infty =$ best approximation to f on $[0, \infty)$ where $p_\infty \in K[0, \infty)$,

$b^* = \inf\{b \in R : \lambda_b = \lambda_\infty\}$,

$$b^{**} = \sup \left\{ b \in \mathbb{R} : \left| f(b) - \frac{1}{p_\infty(b)} \right| = \lambda_\infty \right\},$$

and

$$b_X^* = \inf\{b \in \mathbb{R}: \lambda_b^X = \lambda_\infty^X\}.$$

Observe that $0 < b^* \leqslant b^{**} < \infty$, $\lambda_{b^*} = \lambda_\infty = \lambda_{b^{**}}$, $\lambda_{b_X^*}^X = \lambda_\infty^X \leqslant \lambda_\infty$.

THEOREM 7. *Let* $f \in C_0^+[0, \infty)$, $1/f \notin K[0, \infty)$, $n \geqslant 1$. *Suppose* $X \subseteq [0, \infty)$ *with* $|X| < \delta$ *for some* $\delta > 0$. *Then*

(i) *For any* $\varepsilon > 0$, $b_X^* \in (b^* - \varepsilon, b^{**} + \varepsilon)$, *for all* $\delta > 0$ *sufficiently small. (Thus, if* $b^* = b^{**}$, *then* $b_X^* \to b^*$ *as* $\delta \to 0$.)

(ii) *For every* $\delta > 0$, *sufficiently small, there exists a constant* γ *independent of* X *such that*

$$\left\| f - \frac{1}{p_\infty^X} \right\|_{[0,\infty)} - \left\| f - \frac{1}{p_\infty} \right\|_{[0,\infty)} \leqslant \omega(\delta) + \gamma\delta,$$

where $\omega(\delta) = \max_{x, y \in [0,\infty), |x-y| \leqslant \delta} |f(x) - f(y)|$.

(iii) $1/p_\infty^X$ *converges uniformly to* $1/p_\infty$ *on* $[0, \infty)$ *as* $\delta \to 0$.

Proof. (i) (by contradiction) Suppose there exist sets $\{X_i\}_{i=1}^\infty$ with $|X_i| < \delta_i$, $\delta_i \to 0$ and $b_{X_i}^* \notin (b^* - \varepsilon,\ b^{**} + \varepsilon)$ for some $\varepsilon > 0$ fixed. For notational convenience, let $p_i = p_{b_{X_i}^*}^{X_i}$, $b_i^* = b_{X_i}^*$ and $\lambda_\infty^i = \lambda_\infty^{X_i}$ so that p_i gives the best reciprocal approximation to f on $[0, b_i^*] \cap X_i$ and X_i from $K(X_i)$. If $p_i = \sum_{l=0}^n a_{li} x^l$ then by arguments similar to those of Theorem 1 we have that $\{a_{li}\}$ is bounded, so going to further subsequences, if necessary, we have that $a_{li} \to a_l$ as $i \to \infty$ for $0 \leqslant l \leqslant n$. Set $p(x) = \sum_{l=0}^n a_l x^l$. Again, using arguments as in Theorem 1, it can be show that $p \equiv p_\infty$. Thus, p is not a constant so choosing a nonzero coefficient a_k with $k \geqslant 1$ we must have $a_{ki} \geqslant a_k/2$ for $i \geqslant i_1$ (say) implying there exists $c \geqslant b^*$ such that $1/p_i(x) \leqslant \lambda_\infty/2$ and $f(x) \leqslant \lambda_\infty/2$ for all $x \geqslant c$. By the uniform convergence of $\{p_i\}$ to p_∞ on $[0, b^*]$ and the assumption that $|X_i| \to 0$ it follows that for i sufficiently large $\lambda_\infty^i \geqslant \frac{3}{4}\lambda_\infty$. Thus, for i sufficiently large we have that $b_i^* \leqslant c$. Therefore, $\{b_i^*\}$ is bounded.

Choose a subsequence (note relabelled) so that $b_i^* \to b$ (say), and choose i_2 so large that $b_i^* \in [0, L]$ for all $i \geqslant i_2$, where $L = \max(b^{**} + \varepsilon, b) + 1$. Then

$$\inf_{p \in K[0,\infty)} \left\{ \left\| f - \frac{1}{p} \right\|_{[0,L]} \right\} = \lambda_\infty \text{ and } \inf_{p \in K(X_i)} \left\{ \left\| f - \frac{1}{p} \right\|_{[0,L] \cap X_i} \right\} = \lambda_\infty^i,\ i \geqslant i_2.$$

Now, by the uniform convergence of $\{p_i\}$ to p_∞ on $[0, L]$ we have that $\lambda_\infty^i \to \lambda_\infty$ as $i \to \infty$.

Now suppose $b \geqslant b^{**} + \varepsilon$. Then, we must have that $|f(b) - 1/p_\infty(b)| < \lambda_\infty$

by the definition of b^{**}. Thus, there exists $\eta > 0$ such that for all i sufficiently large, $|f(y) - 1/p_i(y)| < \lambda_\infty^i$, $\forall y \in (b - \eta, b + \eta) \cap X_i$, contradicting the fact that $b_i^* \in (b - \eta, b + \eta) \cap X_i$ for all i sufficiently large.

On the other hand, suppose $b \leqslant b^* - \varepsilon$. Then, by the definition of b^* we have that

$$\alpha = \inf \left\{ \left\| f - \frac{1}{p} \right\|_{[0, b + \varepsilon/2]} : p \in K[0, \infty) \right\} < \lambda_\infty .$$

However, this implies $\lambda_i \leqslant \alpha$ for all i sufficiently large (so that $b_i^* \leqslant b + \varepsilon/2$) which contradicts the fact that $\lambda_i \to \lambda_\infty$. This contradiction then proves part (i) of the theorem.

For parts (ii) and (iii), since $b_X^* \in [0, b^{**} + 1]$ for all $\delta > 0$ sufficiently small, we have

$$\lambda_\infty = \left\| f - \frac{1}{p_\infty} \right\|_{[0, b^{**} + 1]}, \qquad \lambda_\infty^X = \left\| f - \frac{1}{p_\infty^X} \right\|_{[0, b^{**} + 1] \cap X}.$$

Parts (ii) and (iii) then follow since the coefficients of p_∞^X are bounded and p_∞^X is bounded away from zero on $[0, b^{**} + 1]$ so that arguments similar to those in [3, pp. 84–88] can be applied. ∎

We give the following example.

EXAMPLE. Define $f(x) = 1/(x + 1) + g(x)$, where

$$g(x) = \begin{cases} 3/16, & x = 0, \\ -3/16, & x = 1, \\ 3/16, & x = 2, \\ -3/16, & x = 3, \\ 0, & x \geqslant 4, \end{cases}$$

and $g(x)$ is linear in $[0, 1]$, $[1, 2]$, $[2, 3]$ and $[3, 4]$, so $f(x) \in C_0^+[0, \infty)$. Let $n = 1$. Then $1/p_\infty = 1/(x + 1)$, $\lambda_\infty = 3/16$, $b^* = 2$, $b^{**} = 3$.

(a) If $X_i = [0, \infty) \backslash (3 - 1/2i, 3 + 1/2i)$, $i \geqslant 1$, we have $1/p_\infty^{X_i} = 1/(x + 1)$, $b_{X_i}^* = 2$, for all i.

(b) If $X_i = [0, \infty) \backslash [0, 1/2i)$, $i \geqslant 1$, we have $1/p_\infty^{X_i} = 1/(x + 1)$; $b_{X_i}^* = 3$, for all i.

Using other choices of X_i, we can make $b_{X_i}^* < 2$ or $b_{X_i}^* > 3$.

7. NUMERICAL EXAMPLES

We show here some examples which were run on a CDC Cyber 172 in single precision (approximately 15 digits of accuracy). The program used was a combined First Remes-differential correction algorithm program (see [5–7]) with minor changes in two subroutines to force $0 \leqslant q_j \leqslant 1$ instead of $-1 \leqslant q_j \leqslant 1$. The computed approximations of the form $p_1/(q_0 + q_1 x + \cdots + q_n x^n)$ were then normalized by dividing all coefficients by p_1.

EXAMPLE 1. Let $f(x) = ((x+1)/2)\, e^{(1-x)/2}$ and $n = 2$. This function has a maximum at $x = 1$, with $f(1) = 1$ (not the type of function that should be approximated by this sort of theory, in general). Let $X = \{0.01l\colon 0 \leqslant l < \infty\}$. Taking $b1 = 1$, the computed approximation on $[0, 1] \cap X$ (101 points) is

$$\frac{1}{p_1(x)} = \frac{1}{1.09627448},$$

with error norm $\lambda = 0.08781968$ and alternant $\{e_0^2, e_0^1\} \cup \{e_0, e_1\}$ (in particular, $p_1''(0) = 0$, $p_1'(0) = 0$, $f(0) = 1/p_1(0) = -0.08781968$, $f(1) - 1/p_1(1) = 0.08781968$). Now using Newton's method to approximate a solution of $f(x) - 0.08781968 = 0$, we get $x = 9.105$ and $f(9.11) = 0.08763095$. Since f is decreasing for $x \geqslant 1$, we take $b2 = 9.11$. The computed approximation on $[0, 9.11] \cap X$ is

$$\frac{1}{p_{9.11}(x)} = \frac{1}{1.06281016 + 0.04620946x^2}$$

with error norm $\lambda_{9.11} = 0.11654117$ and alternant $\{e_0^1\} \cup \{e_0, e_{1.55}, e_{9.11}\}$. This is not best on $[0, \infty)$, since $f(9.12) - 1/p_{9.11}(9.12) = -0.11654154$. We observe that $p_{9.11}$ is not a constant, so searching for $b3$ (which will be the required b here) with $1/p_{9.11}(b3) \leqslant \lambda_{9.11}$ (which can be done by solving $1/p_{9.11}(b3) = \lambda_{9.11}$, the solution is 12.755), we take $b3 = 12.76$. The computed approximation on $[0, 12.76] \cap X$ is

$$\frac{1}{p_{12.76}(x)} = \frac{1}{1.06281009 + 0.04620952x^2}$$

with error norm $\lambda_{12.76} = 0.11654123$ and alternant $\{e_0^1\} \cup \{e_0, e_{1.55}, e_{9.12}\}$; this is best on X. By comparison, if we remove the nonnegativity restriction on the denominator coefficients, the best computed approximation on X is $(1.21587901 - 0.33317116x + 0.12629914x^2)^{-1}$ with error norm 0.03835538, achieved at the extreme points 0.44^+, 1.97^-, 4.62^+, 11.92^-, where the sign indicates the sign of $f - 1/p$.

EXAMPLE 2. Let $f(x) = (\ln(x+2))^{-1}$, $n = 2$. We first tried $X = \{0.01l\colon l$ integer, $0 \leqslant l < \infty\}$ as above; the computed approximation on $[0, 1] \cap X$ was $(0.69955039 + 0.41523483x)^{-1}$ with eror norm 0.01320544 and alternant $\{e_0^2\} \cup \{e_0, e_{.35}, e_1\}$. Solving $f(x) = 0.01320544$ we got

$x \approx e^{75.7} - 2 \approx 7.52 \times 10^{32}$ which is too large for practical computation. Replacing $b1$ by 100, and replacing X by $X' = \{l\colon l \text{ integer}, 0 \leqslant l < \infty\}$ to save computer time, our computed approximation on $[0, 100] \cap X'$ was $(6.78068253 + 0.17583824x)^{-1}$ with error norm 0.16176462 and alternant $\{e_0^2\} \cup \{e_0, e_2, e_{100}\}$. Solving $f(x) = 0.16176462$ yielded $x \approx 481.9$; the computed approximation on $[0, 482] \cap X'$ was $(0.78105035 + 0.17526786x)^{-1}$ with error norm 0.16236785 and alternant $\{e_0^2\} \cup \{e_0, e_2, e_{128}\}$. This is the best approximation on X'. Having found an approximate location for b^*, we refined the approximation using $[0, 130] \cap X$ (13,001 points); the computed approximation after 22.4 second execution time was $(0.78109464 + 0.17557370x)^{-1}$ with error norm 0.16244044 and alternant $\{e_0^2\} \cup \{e_0, e_{1.84}, e_{128}\}$. This we verified to be best on X by directly checking the error on $[0, 469.59)$ and noting that $f(x)$, $1/p(x) < 0.16244044$ for $x > 469.59$. By comparison, removing the nonnegativity restriction on the denominator coefficients yielded $(0.75913982 + 0.21799463x - 0.00154261x^2)$ as the best approximation on $[0, 130) \cap X$, with error norm 0.12541465 achieved at the extreme points 0^+, 1.50^-, 44.82^+, 130^-. This is not best on X^* due to pole near 144.72.

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